

Thank you for your suggestion and understanding. We will include a brief theoretical analysis of the second-order temporal accuracy in the appendix.

The IMEX Runge–Kutta (IMEXRK) scheme used in this work are shown in Eq.(24)-(27):

$$\frac{dy}{dt} = f^{\text{EX}}(y) + f^{\text{IM}}(y), \quad (24)$$

$$y^{(1)} = y^{(n)} + \Delta t f^{\text{EX}}(y^{(n)}), \quad (25)$$

$$y^{(2)} = y^{(n)} + \frac{1}{2}\Delta t \left(f^{\text{EX}}(y^{(n)}) + f^{\text{EX}}(y^{(1)}) \right), \quad (26)$$

$$y^{(n+1)} = y^{(2)} + \Delta t f^{\text{IM}}(y^{(n+1)}). \quad (27)$$

For explicit term, we first expand $f^{\text{EX}}(y^{(1)})$:

$$f^{\text{EX}}(y^{(1)}) = f^{\text{EX}}(y^{(n)} + \Delta t f^{\text{EX}}(y^{(n)})) = f^{\text{EX}}(y^{(n)}) + \Delta t f^{(\text{EX})'}(y^{(n)}) f^{\text{EX}}(y^{(n)}) + O(\Delta t^2). \quad (\text{B1})$$

Substituting into (26), we obtain:

$$y^{(2)} = y^{(n)} + \Delta t f^{\text{EX}}(y^{(n)}) + \frac{\Delta t^2}{2} f^{(\text{EX})'}(y^{(n)}) f^{\text{EX}}(y^{(n)}) + O(\Delta t^3). \quad (\text{B2})$$

The implicit term can be written via a fixed-point expansion:

$$y^{(n+1)} = y^{(2)} + \Delta t f^{\text{IM}}(y^{(2)}) + O(\Delta t^2). \quad (\text{B3})$$

Substituting (B2) into (B3), we obtain:

$$\begin{aligned} y^{(n+1)} &= y^{(n)} + \Delta t f^{\text{EX}}(y^{(n)}) + \frac{\Delta t^2}{2} f^{(\text{EX})'}(y^{(n)}) f^{\text{EX}}(y^{(n)}) + O(\Delta t^3) \\ &\quad + \Delta t f^{\text{IM}}(y^{(2)}) + O(\Delta t^2). \end{aligned} \quad (\text{B4})$$

Then, we expand $f^{\text{IM}}(y^{(2)})$ and it:

$$f^{\text{IM}}(y^{(2)}) = f^{\text{IM}}(y^{(n)}) + \frac{\Delta t}{2} f^{(\text{IM})'}(y^{(n)}) f^{\text{IM}}(y^{(n)}) + O(\Delta t^2). \quad (\text{B5})$$

We can find that $f^{\text{IM}}(y^{(2)})$ is first-order and $\Delta t f^{\text{IM}}(y^{(2)})$ is second-order. Substituting (B5) into (B4), we obtain:

$$\begin{aligned} y^{(n+1)} &= y^{(n)} + \Delta t f^{\text{EX}}(y^{(n)}) + \frac{\Delta t^2}{2} f^{(\text{EX})'}(y^{(n)}) f^{\text{EX}}(y^{(n)}) + O(\Delta t^3) \\ &\quad + \Delta t \left(f^{\text{IM}}(y^{(n)}) + \frac{\Delta t}{2} f^{(\text{IM})'}(y^{(n)}) f^{\text{IM}}(y^{(n)}) \right) + O(\Delta t^3). \end{aligned} \quad (\text{B6})$$

By rearranging the terms, we obtain (B7), which matches the Taylor expansion

of the exact solution to second order.

$$\begin{aligned} \mathbf{y}^{(n+1)} = & \mathbf{y}^{(n)} + \Delta \left(t \mathbf{f}^{\text{EX}}(\mathbf{y}^{(n)}) + \mathbf{f}^{\text{IM}}(\mathbf{y}^{(n)}) \right) \\ & + \frac{\Delta t^2}{2} \left(\mathbf{f}^{(\text{EX})'}(\mathbf{y}^{(n)}) \mathbf{f}^{\text{EX}}(\mathbf{y}^{(n)}) + \mathbf{f}^{(\text{IM})'}(\mathbf{y}^{(n)}) \mathbf{f}^{\text{IM}}(\mathbf{y}^{(n)}) \right) + O(\Delta t^3). \end{aligned} \tag{B7}$$

This Hence, the scheme is formally second-order accurate in time, despite the use of a first-order implicit method.